What is Normal Anyway? Therapy for Epistemological Anxiety

Uri Wilensky
Center for Connected Learning
Northwestern University
Annenberg Hall 311
2120 Campus Drive
Evanston, IL 60208
uriw@media.mit.edu
847-467-3818

1.0 Introduction

Concepts and intuitions from probability and statistics pervade many aspects of our "non-mathematical" lives. When we decide whether to pay more for a car that is reported to be "more reliable"; when we choose to safeguard our children by living in a "low-crime" neighborhood; when we change our diet and exercise habits to lower the risk of heart disease; when we accept or refuse a job with other options pending; all the above actions involve implicit probability judgments. Arguably, no other area of mathematics can be said to apply so widely. No other area is so potentially empowering- so useful in making decisions, both intimate and professional, and affords us such opportunities for making sense of phenomena, both natural and social.

Yet, as the research literature (discussed below) has made abundantly clear, students do not take great advantage of these powerful ways of making sense of the world. This paper seeks to give an account of this missed opportunity in terms of the framework of Connected Mathematics (Wilensky, 1993).The paper begins with a sketch of background issues related to learning probability and statistics. The Connected Probability Project (Wilensky, 1993; 1995a; 1995b) is then described and situated in the framework of Connected Mathematics. The notion of "epistemological anxiety" as a primary obstacle to learning probability and statistics is articulated and developed. The paper then describes how the use of computational environments -- specifically object-based parallel modeling languages -- can allow learners to successfully address epistemological anxiety in the realm of probability and statistics. The remainder of the paper seeks to instantiate the theory and practice of Connected Mathematics in case studies concerning learning about the concept of normal distribution. These cases illustrate how traditional teaching of the concept of normal distribution which relies on formalism and macro- level summary statistics leads to epistemological anxiety. The cases also illustrate how Connected Mathematics engages issues of epistemological anxiety and how it fosters a deeper understanding of normal distributions. This understanding is advanced through learners modeling normal distributions as emergent phenomena.

1.1 The Subject Learners are Required to Hate

People's attitudes towards probability and statistics can be summed up in the well worn adage attributed to both Twain and Disraeli: "There are three kinds of lies: lies, damn lies, and statistics." (in Tripp, 1970)

Probability & Statistics (henceforward, P&S) research "methods" courses have achieved notoriety in the undergraduate and graduate curriculum as the bane of many a student. Students in the social sciences are usually required to take a statistics course. Though such courses are intended to introduce them to concepts and methods which they can then employ in making sense of real data, they frequently report being able to work the textbook problems but having little idea of "what they mean," or how to apply them to novel situations (see, e.g., Phillips, 1988; {Rubin et al, 198x}). Students complain about being able to make several different arguments to solve a problem, all of which sound plausible yet yield different answers (Wilensky, 1993). How are they to choose among the plausible alternatives?

This confusion and indecision leads to what I have termed "epistemological anxiety" (Wilensky, 1993)- a feeling, often in the background, that one does not comprehend the meanings, purposes, source or legitimacy of the mathematical objects one is manipulating and using.

1.2 Responding to a literature of Innate Constraints

There has been considerable research (e.g., Gould, 1991; Konold, 1991; Phillips, 1988; Piaget, 1975; Tversky & Kahneman, 1974) documenting difficulties people have in learning, understanding and using concepts of probability and statistics. Much of this research locates the source of the problem in cognitive constraints of the mind of the learner (see e.g., Cohen, 1979; Edwards & von Winterfeldt, 1986; Evans, 1993; Gould, 1991; Kahneman & Tversky, 1973; 1982; Nisbett, 1980; Nisbett et al, 1983; Tversky & Kahneman, 1974; 1980; 1983).

In their now classic work, Tversky & Kahneman (1982) document the persistent errors and "misconceptions" that people make when making probabilistic "judgments under uncertainty". Among these errors are systematic misconceptions about probabilities of conjunctions, inverse probabilities, updating probabilities in the face of new evidence and "seeing" non-random pattern in random data. These systematic errors are repeatable and don't seem to go away even when people have had significant training in probability. This contributes to a widespread belief that humans are incapable of thinking intuitively about probability.

Tversky & Kahneman speculate as to the origin of these systematic biases in people's assessment of likelihoods. They theorize that people's mental resources are too limited to be able to generate probabilistically accurate judgments. Consequently, people are forced to fall back on computationally simpler "heuristics". In other words, we have been "hard-wired" not be able to think about probability and must circumvent our natural thinking processes in order to overcome this liability.

The view that humans must circumvent natural thinking processes in the realm of probability has spawned a large literature and has become very influential. For the purposes of this exposition, let us call this view the "accommodationist" view. The effect of this accommodationist view on P&S instruction has been a deference to the mechanical manipulation of the tokens of formal notation as a guard against unreliable intuitions. This lesson for P&S instruction has reached the highest levels of education. In an introductory graduate course in probability and statistics at a major research university, the professor wrote down Bayes theorem for calculating inverse probabilities and then baldly announced:

"Don't even try to do inverse probabilities in your head. Always use Bayes formula. As Tversky and Kahneman have shown, it is impossible for humans to get an intuitive feel for inverse probabilities". (in Wilensky, 1993).

This quote is a pedagogical embodiment of the accomodationist view.

It is possible to begin with a more conservative set of assumptions then do the accommodationists. We could

assume that P&S is an area of mathematics not widely separated from the rest of the discipline. Furthermore, we could posit a broadly developmental view of mathematical intuition which allows for conceptual change. We would then look elsewhere for the source of learners' difficulties with P&S. We would look at the learning methods, tools and cognitive technologies (Pea, 1987) available to the learner. Given the prevalence of this accomodationist view, it would be of great interest if it could be shown that a suitable learning environment could help learners transform their probabilistic biases into effective intuitions and a solid conceptual understanding of P&S. This work is being undertaken through the Connected Probability project.

2.0 Connected Probability

The mission of the Connected Probability project (Wilensky, 1993; 1995-a; 1995-b) is to better understand the source of learners' difficulties in P&S and to build learning environments that would foster the development of intuitive conceptions of foundational concepts and positive attitude towards the discipline. The Connected Probability project was launched with the hypothesis that learners' difficulties with P&S are not primarily due to hard-wired cognitive constraints. We conjecture that students' antipathy towards P&S is primarily due to epistemological anxiety concerning the basic concepts of the discipline. We further locate the source of the anxiety in the lack of sufficiently powerful learning environments for P&S. The paucity of the learning environment stems, in great part, from the unavailability of powerful tools for experimentation and construction in probability. The anxiety is further reinforced by the teaching practices employed in mathematics classrooms and a "protective" culture which discourages revelation of mathematical conceptions and process and its consequences for mathematical discourse in the classroom. Given this "diagnosis", it follows that learners' difficulties in P&S can be effectively addressed without resorting to either rigidly formalized instruction or neural surgery.

In the initial year of the project, seventeen in depth interviews about P&S were conducted with learners age fourteen to sixty-four. Interviews were open ended and most often experienced by the interviewees as extended conversations. The interviewer guided these conversations so that the majority of a list of 23 topics was addressed. The interview topics ranged from attitudes toward situations of uncertainty, to interpretation of newspaper statistics, to the design of studies for collecting desired statistics and to formal probability problems. In most interviews, a computational modeling environment designed for experimenting with P&S was made available to the learners.

2.1 Probability Distributions

The interview format was designed to focus on several key concepts in P&S. The notion of Probability distribution is one such key concept. We focus on probability distribution, partly because of its importance: 1) an understanding of probability distribution is crucial to understanding the statistical models ubiquitous in scientific research. 2) The concept is equally central to participation in the public forum as an informed citizen. 3) Without the concept of distribution learners cannot truly understand how events can be both unpredictable and constrained - we cannot have a coherent concept of randomness (Wilensky, 1993; 1995-b). 4) Probability distributions stand at the interface between the traditional study of probability and the traditional study of statistics and, thus, afford an opportunity to make strong connections between the two disciplines. Another reason we chose to focus on probability distribution is the potential for bringing about meaningful change in the learning experience for many students. In a typical course in probability and statistics, students are exposed to a standard library of distributions and associated formulae, but do not have a chance to construct these distributions and understand what "problems they are trying to solve". The availability of new computational "object-based parallel modeling languages" (Wilensky, forthcoming) affords learners the ability to construct these distributions as patterns emergent from probabilistic rules. Through these constructions, learners can make connections between probabilistic descriptions of discrete phenomena and the statistical descriptions of the ensemble -- in the process seeing the utility of both descriptions. Meaningful intervention is possible.

An analysis of the interview data revealed persistent confusions on the part of most interviewees about probability distributions. These confusions and attendant anxieties resulted in the interviewees turning off their sense-making with regard to probability distributions and merely invoking rote formulae when applying probability distributions to real world problems.

3.0 Connected Mathematics

Central themes from the Connected Mathematics research program.(Wilensky, 1993; 1995c) inform the Connected Probability project. The Connected Mathematics program takes broad aim at both our view of the discipline of mathematics and the practice of mathematics pedagogy. Connected Mathematics is situated in the Constructionist learning paradigm (Papert, 1991; 1993). The name Connected Mathematics comes from two seemingly disparate sources, the literature of emergent Artificial Intelligence (AI) and the literature of feminist critique. From emergent AI and, in particular, from the Society of Mind theory (Minsky, 1987; Papert, 1980), Connected Mathematics takes the idea that a concept cannot be intelligible if it has only one meaning, it is through connections that concepts gain meaning. The feminist literature (e.g., Belenky et al., 1986; Keller, 1983; Gilligan, 1977; Surrey, 1991), contributes the idea of "connected knowing", a personal form of knowing that is intimate and contextualized as opposed to an alienated and disconnected formal knowing. Mathematical concepts derive their meaning and their power through their embeddedness in a personally and socially constructed web of connections to other ideas and experiences, both mathematical and non-mathematical.

In a Connected Mathematics learning environment, the focus is on learner-owned investigative activities followed by reflection. Thus, students are not led through the mathematical "litany" of definition, theorem proof. Mathematical concepts are not simply given by formal definitions. Instead, mathematical concepts are multiply represented (e.g., Kaput, 1987 ; Von Glaserfeld, 1989) and the environment supports multiple styles and ways of knowing (see e.g., Turkle & Papert, 1991). Mathematical intuitions are not assumed to be static, nor are some mathematical concepts assumed to be "abstract" and thus not amenable to intuitive apprehension. Learners are supported in building and developing their mathematical intuitions over a lifetime and, through this construction process, mathematical objects are seen to be more concrete (Wilensky, 1991; 1993) as learning progresses.

Connected Mathematics views mathematics as something human make ã we build tools to operate upon and make sense of our world . In contrast to learning procedures or formalisms first, as in the traditional curriculum, Connected Mathematics calls for making many more connections between mathematics and the world at large as well as between different mathematical domains throughout the learning experience(e.g. Cuoco & Goldenberg, 1995). Empowering technology is central to this effort. Technology is used as a personally expressive medium - to explore areas of mathematics previously inaccessible, to make abstract mathematical concepts concrete, and to create new mathematics In contrast to reform documents such as the NCTM standards, which portrays an "image" of mathematics as essentially a problem solving activity, its vision of mathematics is a more generative one - the central activity being making new mathematics. A culture of design and critique is developed.

Connected Mathematics pays great attention to the affective side of learning and doing mathematics. As such, it pays attention to the role of play, joy wonder and curiosity in the learning of mathematics and to the role of desire for social connection as a motivation for engaging in mathematical activity (see also Thurston, 1992). Difficulties in learning mathematics are also examined from an affective perspective. Connected Mathematics looks critically at the role of shame in mathematical culture -- how shame creates, first, anxiety (see e.g., Chipman, Krantz & Silver, 1994; Steele, 1995; Tobias, 1993) and, ultimately, negative mathematical self-image (Dweck & Leggett, 1988; Steele, 1995) in the individual learner and how it stifles discourse and enforces hierarchy (O'Connor, 1993) in the mathematical community. In contrast, a Connected Mathematics learning environment fosters an atmosphere in which it is safe for mathematical learners to express their partial understandings and values these understandings regardless of their degree of correspondence with the

mathematics canon. In doing so, it parts company with the literature on misconceptions which highlights the gulf between expert and novice. Connected Mathematics, instead, stresses the continuity between expert and novice understanding. This continuity obtains both in their explicit conceptions - novice conceptions are not discarded on the road to expertise but rather repurposed, and in their general "messy" character -- even for expert mathematicians, only small areas of clarity have been laboriously carved out from the generally messy terrain. (see also Smith, diSessa & Roschelle, 1994).

4.0 Sources of Epistemological Anxiety and the Making of Mathematical Culture

As discussed above, learners of probability suffer from an attendant anxiety about the source of legitimacy of the probabilistic procedures and formulae they are learning. This anxiety is not unique to P&S. Indeed, it is quite common in mathematics instruction.

The sources of this anxiety are multiple. The culture of the mathematics classroom contributes greatly (Hoyles, 1985; Lampert, 1991; Noss, 1988; O'Connor, 1993). Its insistence on unitary definitions for mathematical concepts reinforces the belief that mathematical knowledge is completely disconnected from the rest of the learner's knowledge (Cuoco, Goldenberg & Mark 1995; Minsky, 1987-b; Wilensky, 1993). The learner , thus poised at the lip of a yawning chasm separating her previous conceptions from the new knowledge, is understandably anxious. The formal definitions sever the links to the many related conceptions that could serve as bridging mechanisms and could alleviate anxiety. The anxiety is further enhanced by social isolation. The insistence on answers or results, absent of intellectual texture, and invalidation of personal voice (Confrey, in press; Gilligan, 1977) discourages the learner from expressing her partial and incomplete understanding. This, in turn, can create the sense that everyone else is clear about the meaning of these objects and that you alone are confused. The further knowledge that any admission of confusion will be used to rank you in the mathematics hierarchy and judge your mathematical intelligence makes it practically taboo to share your "messy" concepts (Papert, 1972; 1993; Wilensky, 1993) and attendant epistemological anxiety.

Connected Mathematics provides several sources of therapy for epistemological anxiety. Instead of formal definitions, it emphasizes connections to the learner's knowledge that make the transition to new knowledge both safer and more meaningful. By exposing the universality of confusion and messy concepts, it reassures the learner that her predicament is "normal" and shared. This, in turn, leads to the voicing of previously unvoiced concepts and sets the context for the social negotiation of mathematical knowing. This emphasis on making, voicing and sharing takes public the activity of understanding mathematics. These changes in mathematical culture are both supported and made much more realizable by the advent of computational modeling environments.

4.1 Epistemological Anxiety in the Realm of Probability

There are several reasons why epistemological anxiety is particularly pronounced in the domain of P&S:

1) We are living in a time when the meanings of the basic notions of probability theory, the ideas of "randomness", "distribution", and even "probability" itself are still quite controversial. There is, as yet, no consensual agreement on the foundational concepts of probability † they are still being debated by mathematicians and philosophers (see. e.g., Cartwright, 1987; Gigerenzer, 1987; Savage, 1954; Suppes, 1984; Von Mises, 1957). There is particular disagreement on the applicability of probability to individual instances. This leads to confusion about the applicability of probability theory to the unique situations in our lives. It raises doubts about its utility in making individual choices which might serve to animate our interest in the subject.

2) When we encounter statistical data (as we commonly do in newspaper articles), it is often detached from its method of collection. If a statistic is left vague and disconnected in this way, we can not operate on it,

transform it and compare it with others so as to make it meaningful and concrete.

3) Short term memory limitations make it difficult to collect large amounts of data in "real time". So instead, we construct summaries of the data. These summaries are not analyzable into their constituent elements and thus necessarily remain disconnected from the original data. In this way P&S is at a stage of its development not unlike other areas of mathematics throughout history. These same kinds of memory constraints once made arithmetic very difficult until memory saving technologies (such as zero, place value and generalized decimal notation) were invented (See e.g., Kline, 1972; McLuhan, 1964; Rotman, 1987). A central conjecture of the Connected Probability project is that computational technology should do for probability what Hindu-Arabic notation has done for arithmetic. Computational environments allow large bodies of data to be visualized at once in a small space and large numbers of repetitions to happen in a short time. As a result, short term memory resource limitations can be overcome. The focus can then be on probabilistic reasoning.

4) It is very hard to get feedback as to the adequacy of our probability judgments. If we assess that the probability is 30% that it will rain tomorrow and it does rain, what have we learned? We require many, many such observations in order to calibrate our judgments to the data. Again, memory limitations block us from receiving the necessary feedback, Moreover, there is little opportunity for control because the world is constantly changing and we cannot repeat the experiments so as to get systematic and controlled feedback 5) Seeing our lives as experiments in probability requires seeing our current situation as one of a large collection of similar possible situations. To make this large collection into a "concrete" object of thought (Wilensky, 1991) requires a massive construction job. It requires forging links between our current situations and situations forgotten in the past, situations not yet arisen, and counterfactual situations, (those situations which could have arisen but did not) .

6) A powerful way to think about probability situations is to think of multiple, interacting, distributed entities. Again we cannot keep so many entities in working memory (Case, 1993; Miller, 1956) at once. Consequently, we construct wholes out of the many interacting parts and understand the behavior of the ensemble in terms of its average behavior.

Memory and resource limitations, absent a notation or medium, do contribute to why, until now, many people have not developed robust probabilistic intuitions. We have reached a time where these limitations will no longer hold sway in P&S. The emergence of computational modeling environments promises to provide a holding environment , a symbolizing medium which can express data distributed over space and time. This should enable us to see view dynamic properties of ensembles and to conduct experiments with immediate feedback. It is for this reason that one of the most powerful forms of therapy for epistemological anxiety is access to a computational environment in which the objects whose epistemological status are in doubt can be modeled, experimented with and debugged to the satisfaction of the learner. The computer serves as a setting and symbolizing medium for building mathematical intuitions.

5.0 Computational Modeling in Mathematics and Science Education

In a computational modeling approach to mathematics and science education, a modeling language and sets of associated tools are made available to learners. Learners then choose a concept or phenomenon and create a computational model of the phenomenon. In contrast to pre-built simulations, where the learner is interacting with an expert model, the model-building approach allows the learner to own and pursue personally meaningful investigations. In a model building approach, there are no "black boxes" in the phenomenon of interest - the learner constructs her own "boxes".

The modeling approach has both cognitive and affective benefits. By building computational models of everyday and scientific phenomena, learners can develop robust mental models of the underlying probability and statistics. The feedback provided by building and then testing the computational model supports the learner

in debugging and successively refining (Leron, 1994) the mental model. On the affective side, the modeling environment supports the learner's expression of partial understanding. In a modeling environment, it is clear that partial understandings are the norm -- one builds models by, initially, expressing a tentative model in the modeling language, then gradually refining and debugging the model. In this way modeling activities short circuit the "whole cloth" answer orientation of the culture of classroom mathematics (Sfard & Leron, in preparation). Indeed, the activity of model building is so different from normal mathematics for most students that they, often, do not identify themselves to be "doing mathematics". ({Feurzeig, 19xx; Mandinach & Cline, 1994; Thornton, 19xx}). Embedded in a suitable social environment, the opportunity to express partial understandings without judgment can go a long way towards alleviating a sense of mathematical shame. Relieved from the necessity of "getting it right" the first time, even learners not usually considered good at mathematics and science can build models that demonstrate a qualitatively greater level of mathematical achievement than is usually found in mathematics classrooms. Computational modeling serves as a powerful therapy for epistemological anxiety.

5.1 Modeling Emergent Phenomena

One productive domain for computer modeling is "emergent phenomena", in which global patterns emerge from local interactions. Emergent phenomena can provide rich contexts for learners to build with probabilistic parts. Learners can explore the stable structures that emerge when probabilistic behavior is given to distributed computational agents. Thus, instead of encountering probability through "solving" sets of decontextualized combinatoric problems, learners can participate in constructionist activities - they design and build with probability.

Probability distributions can be seen as canonical cases of emergent phenomena. They are stable global structures that arise from the interactions of multiple distributed agents. Typically, in statistics classes, the emergent nature of distributions such as the normal distribution (or "bell curve") is hidden. We "learn" about distributions through descriptions of their global characteristics (e.g., mean, standard deviation, variance, skew, moments). This conceals the way these distributions are built up or emerge from individual instances. As a result, the critical connection between the micro- level of the phenomenon and the macro- level is severed. The learner is given the macro- description and formulae and is asked to accept on authority that these macro-descriptions are appropriate for such and such a set of phenomena. This supplanting of experimentation with external authority contributes significantly to epistemological anxiety.

5.2 Object-based Parallel Modeling

The modeling environment used in the Connected Probability project, is the language StarLogo (Resnick, 1992; 1994, Wilensky, 1993; 1995b) extended to be especially useful for modeling phenomena in P&S. StarLogo is an extension of the Computer language Logo. In Logo, a graphical turtle is controlled by issuing movement commands. In StarLogo, however, the user can control thousands of "turtles" or "agents". StarLogo is an example of a new kind of modeling language ã it is an object-based parallel modeling language. Other examples of such languages include KidSim (Smith, Cypher & Spohrer, 1994) and AgentSheets (Repenning, 1993). "Parallel" means that the agents behave as if they are all executing their commands simultaneously. StarLogo was originally implemented on the Connection Machine, a massively parallel supercomputer, and was thus in fact parallel, each agent controlled by a separate processor. In more recent implementations, StarLogo runs on serial machines such as the Macintosh computer, so it is running as a parallel "virtual machine" on top of a serial architecture.

"Object-based" means that each agent is self-contained: it has its own internal state and communicates with other agents primarily by local channels - agents don't do much action at a distance. The computer language Logo had a single such object - the "turtle". Papert (1980) argued that the power of the turtle to facilitate learning geometry lay in the fact that the child could identify with the turtle - enabling "syntonic" learning.

Object-based parallel modeling languages such as StarLogo afford greater identification with their objects, and thus, in contrast to more procedural languages, foster syntonic learning of emergent phenomena.

Learners can assign micro- level rules to simultaneously control thousands of such turtles. Macro- phenomena emerge in relation to these micro- rules. This makes both forwards and backwards modeling possible - that is 1) exploring the global effects of various sets of local rules. and 2) trying to produce a known global phenomenon through finding appropriate local rules (Wilensky, 1995a).

Such object-based parallel modeling languages can be powerful environments for doing P&S. One way to think of these agents is as multiple sided irregular dice. At each step in running a model built in one of these languages, thousands of dice are thrown and results visually displayed. This use of object based parallel modeling languages is reminiscent of the technique of Monte Carlo simulations and shares feature with resampling statistics approaches (Diaconis & Bradley 1983; Konold, 1994; Simon et al, 1976; Simon & Bruce, 1991). Unlike these other approaches, in object-based parallel modeling languages, control of the program is distributed through the agents rather than centrally controlled and the agents themselves are inspectable, visualizable, "concrete" objects.

Even at this basic level of "dice rolling", many experiments in probability and statistics are suggested. Indeed, most of the standard elementary curriculum in P&S can be easily modeled in StarLogo using these "dice-rolling" features of the language alone. In a non-computational setting, experiments with large data sets are perforce constrained to be summarized by mediating global formulae and statistics. Experiments can be done for small data sets and small numbers of trials and is typically confined to several con-flips, dice-rolls and spinner-spins. But once the scale is increased, the ability to experiment is gone. As discussed earlier, the critical connection between the micro- level of the phenomenon and the macro- level is severed.

Additionally, in object-based parallel modeling languages, the "dice" can also interact with each other. As a result, many complex interactions are possible - the results of which are not predictable using standard mathematical models. The ability to experiment with such complex interactions and get visual feedback however, allows learners to make qualitative sense of different possible patterns, classifying meta-patterns and noting trends such as feedback cycles, critical densities, clustering, etc. Even though these agents -- thought of as dice -- are probabilistic components whose individual states cannot be predicted, the emergent properties of the collection of agents can be stable and predictable. This insight is essential to grasping the key notion of probability distribution. For this reason, object-based parallel modeling languages are excellent constructionist environments for learning probability -- the learner is constructing objects using probabilistic components. This situation is analogous to the acquisition of print literacy as it includes both reading and writing. Instead of just "reading" about probability (by being given the summary formulae), the learner "writes" with probability.

6.0 Case Studies

The remainder of this paper will be devoted to two case studies. The cases, taken from Connected Probability interviews, are offered as illustrations of epistemological anxiety concerning the notion of normal distribution and, in one case, a successful therapy through StarLogo modeling in a Connected Mathematics context.

6.1 Normal Anxiety - Lynn's tricky distributions

Let me introduce Lynn, a psychologist in her mid-thirties who at the time of her interview had recently received her Ph.D. For her dissertation, Lynn had done a statistical comparison study of different treatment modalities for improving the reading comprehension of dyslexic children. While in graduate school, she had taken an introductory probability and statistics class as well as two classes in statistical methods. As such she would "naturally" be classified as having a fairly sophisticated background. In this interview fragment we will see that despite her sophisticated background, basic ideas of randomness and distribution are alienated for her - neither

trusted nor appropriated for her ends. Even though Lynn was capable of applying formal statistical methods in her coursework, she was fundamentally confused about the "madness behind the method". In this interview, she starts to negotiate meaning for "normal distribution". While her attempts take her to a position vis a vis distributions which would be considered just plain wrong in most university courses and may indeed be incoherent, she is for the first time grappling with the meaning of the fundamental ideas of randomness and distribution. In so doing, she takes a step towards developing intuitions about and appropriating these concepts.

As background to the interview, I asked Lynn about her attitudes towards mathematics. Her reply was indicative of the issues that would arise in the body of the interview:

U: So, what was math like for you in school?

L: Well, I was always good at math. But, I didn't really like it.

U: Why was that?

L: Why? I donêt know. I guess I always felt like I was getting away with something, you know, like I was cheating. I could do the problems and I did well on the tests, but I didnêt really know what was going on.

One of the first questions that arose in the body of the interview with Lynn was: What would you estimate is the probability that a woman in the US would be at least 5'5" tall? Here is the text of the ensuing conversation:

L: Well I guess it would be about 1/2.

U: Why do you say that?

L: Well height is normally distributed and I'd say the mean height of women is about 5'5" so half of the women would be taller than 5'5".

U: Why would half the women be taller than the mean?

L: Because the curve of the normal distribution is symmetric around the mean - so half would be below it and half above it.

U: What about the number that are exactly 5'5" tall?

L: Well I guess they could make a difference, but no - they shouldn't matter because they're just one data point and so can be ignored.

U: You can ignore any one data point?

L: Yes, ... because there are infinitely many data points so one doesn't make a difference.

U: But can you make infinitely many height discriminations? How do you measure height?

Here, I'm just trying to probe Lynn's thinking about discrete vs. continuous distributions.

L: Well.... I guess we measure it in inches -- so there probably aren't infinitely many data points. I'm somewhat confused because I know height is distributed normally and I know that for normal distributions the probability is 0.5 of being bigger than the mean, but how come you can ignore the bump in the middle? I guess 0.5 is just an estimate, it's approximately 0.5.

U: How do you know height is distributed normally?

L: I don't remember where I first learned it, but lots of the problems in the stats books tell you that.

My question was intended to elicit an explanation of why Lynn believed height was distributed normally in terms of her personal knowledge, but Lynn responded to it as about probing the context of gaining that knowledge and the authority it derives from.

U: *Why do you think height is distributed normally?*

L: *Come again? (sarcastic)*

U: *Why is it that women's height can be graphed using a normal curve?*

L: *That's a strange question.*

U: *Strange?*

L: *No one's ever asked me that before..... (thinking to herself for a while) I guess there are 2 possible theories: Either it's just a fact about the world, some guy collected a lot of height data and noticed that it fell into a normal shape.....*

U: *Or?*

L: *Or maybe it's just a mathematical trick.*

U: *A trick? How could it be a trick?*

L: *Well... Maybe some mathematician somewhere just concocted this crazy function, you know, and decided to say that height fit it.*

U: *You mean...*

L: *You know the height data could probably be graphed with lots of different functions and the normal curve was just applied to it by this one guy and now everybody has to use his function.*

U: *So youêre saying that in the one case, it's a fact about the world that height is distributed in a certain way, and in the other case, it's a fact about our descriptions but not about height?*

L: *Yeah.*

U: *Well, if you had to commit to one of these theories, which would it be?*

L: *If I had to choose just one?*

U: *Yeah.*

L: *I don't know. That's really interesting. Which theory do I really believe? I guess I've always been uncertain which to believe and it's been there in the background you know, but I don't know. I guess if I had to choose, if I have to choose one, I believe it's a mathematical trick, a mathematician's game. ....What possible reason could there be for height, ...for nature, to follow some weird bizarro function?*

The above was a short section of the first probability interview I conducted . Until the last exchange transcribed, I had the feeling that the interview was not very interesting for Lynn. But after that last exchange she got very animated and involved in the interview. This question of the reality vs. "trickiness" of the normal and other distributions occupied her for much of the next few discussions. She came back to it again and again.

At the time, I was a bit surprised by this reaction. Earlier in the interview, when I asked her about infinitely many height discriminations, I think I was trying to steer Lynn into making a distinction between a discrete and continuous distribution. This, I thought, might have resolved her dilemma about ignoring the "bump in the middle". But I was not expecting to hear such basic doubts about the validity of statistics from someone with so much experience studying and using statistical methods .

Lynn's question was, as she says, "in the background" all the time when she thought about statistics. How could it not be? At the same time that she was solving probability and statistics problems for homework, and later on when she actually implemented a study which used statistical tests to evaluate treatments, she was never really sure that the formulae she was using had any explanatory power, any real meaning other than mathematical conventions. Is it any wonder, then, that she was engaged by this question? Up until now, no one had engaged her in discussion about how to interpret the mathematical formulae she was using. She was left feeling that her own work played by the "official rules" but in a way was not accessible to her -- and she harbored doubts about its validity. Her questions, while essentially epistemological, go to the heart of the mathematics of probability. Note that, in emphasizing the normal distribution, her courses had led her to see the process of describing height by a curve as a single process with a single agent (possibly the mathematician) controlling it. The idea of the height distribution as being emergent from many interacting probabilistic factors was not in awareness.

Had we tested Lynn on textbook statistics problems, she would have performed flawlessly . Yet, even in a interviewee of such sophisticated background, epistemological anxiety was lurking in the background. Lynn knew that she did not really know what gave validity to the procedures she had so laboriously mastered. Through engaging in this interview, her fundamental confusion is exposed. To an uninformed observer, it might seem that the interview has caused Lynn to regress, confusing her about material she has mastered. But, in fact, by expressing her partial understandings of distributions through asking whether normal distributions are "just a mathematician's trick", Lynn takes a big step towards a more Connected Mathematical understanding of distributions.

In the next interview, we will see how another interviewee, engaged in building a StarLogo model of normal distributions, deals with these same issues.

6.2 Modeling Normal Behavior - Alan's Hopping Rabbits

Alan, a graduate student in media studies had a strong college mathematics background. Although, he is mathematically quite sophisticated, Alan also expresses confusion about normal distributions:

A: I never really understood normal distributions. I mean I can do the problems, I've even helped Wendy [his wife] with her statistics problems, but what is really going on? Why should height fall into a normal distribution?

As the interview progressed, Alan became intrigued by the question: "why is height normally distributed?" After some prodding by the interviewer to come up with an answer to his question, Alan made a conjecture as to why height is normally distributed:

A: You start out with this one couple, Adam and Eve say, and they're a certain height. No, make this simpler, we just have Adam and he's a certain height. Now let's suppose we have parthenogenesis, (is that the word?) and Adam has kids on his own. And suppose his kids are a bit off from where he is [that is, slightly different heights] due to copying errors or something. Then they sort of form a little bell curve - anyway a binomial distribution. Now suppose they have children and that process continues many generations, then wouldn't you wind up with a normally distributed population?

To explore this conceptual model, Alan decided to write a StarLogo model with me. The model is initialized to have thousands of "rabbits" at the bottom middle of the graphics screen. (Call the initial location of the rabbits

the "origin". For reference, a green vertical line is placed at the origin.) Rabbits can hop right or left from the origin on a line at the bottom of the screen. (Call it the hopping line) Each rabbit has a set of possible step sizes which it can hop in one clock "tick". Associated with each step size is a probability - the probability that the rabbit will choose that step size. To make the interface easier for himself and other possible users who might not be comfortable with probabilities (expressed as real numbers between 0 and 1), the probabilities of rabbit steps are expressed as integer relative odds. The rabbits are initialized with the command "setup-rabbits" which takes three arguments: the number of rabbits, a list of step-sizes and a list of probability-ratios. So, for example, if one wanted to create 1000 rabbits that could hop to the right 2 units with probability 3/4 and hop to the left 1 unit with probability 1/4, one would type the command

setup-rabbits 1000 [2 -1] [3 1].

After executing this command, the rabbits would be all piled on top of each other at the middle bottom of the screen. It is then possible to perform experiments and let them hop for a while. At each clock cycle each rabbit picks a step to hop according to the probabilities and hops that distance on the hopping line. Once the rabbits have hopped for one clock tick, they are no longer all in the same place. They are still on the hopping line, but at different x positions on that line. Above each position at which there are rabbits present, we display a yellow column - the height of which is proportional to the number of rabbits at that location. In this way an evolving "living" histogram of the distribution of rabbit locations on the hopping line appears on the screen.

To make this a bit clearer, letês take a simple example:

If we initialize the rabbits with the command setup-rabbits 8000 [1 -1] [1 1], then 8000 rabbits will appear at the origin. Each rabbit has been initialized so that it can only hop either one unit to the left or one unit to the right with equal probability. If we let them hop one step, then because the probability of moving left is the same as the probability of moving right, approximately half the rabbits will move one unit left and another half will move one unit right. As a result two roughly equally high yellow columns will appear on the screen above locations x=1 and x=-1. If we now let the rabbits hop one more step, there will be a tall yellow column in the middle and shorter, roughly equally high, yellow columns to the right and to the left. Continuing this process will lead to histograms of binomial distributions symmetric about the origin. (See figure below).

setup-rabbits 8000 [1 -1] [1 1] after 10 "hops"
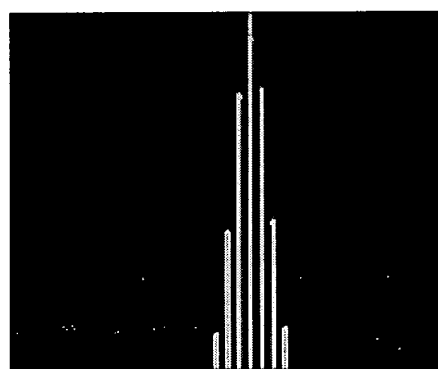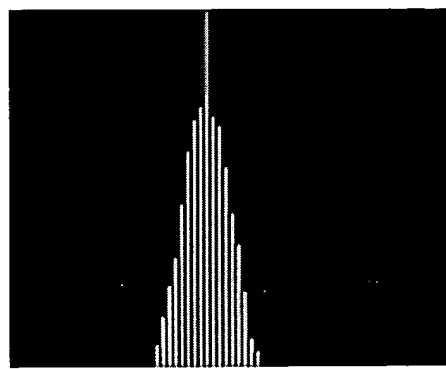
setup-rabbits 8000 [1 -1] [1 1] after 40 "hops"

In the above simple example, when the rabbits hop, their average location does not move from the origin ã the histogram of rabbits is symmetric about the origin. To help visualize the movement of the rabbits when the average location does change, a purple column is displayed above the average location and a green column "remembers" where the rabbits started. Natural questions that arise in this rabbit jumping "microworld", then, are: what will cause the average location of the rabbits to change and what will cause more of them to be one side of the average than on the other?

As you may recall, the reason Alan and I created this model was to investigate Alan's theory of height distributions. One way to think of the rabbits when they're all at the origin is as a population of individuals all of a standard height. When rabbits take a step they represent new individuals in the population who have

deviated slightly from their parents' height. In our simple example the average height of the population does not change and the population heights are symmetric about this average.

When viewing this example, Alan exclaimed: "I see why population heights are symmetrical, the rabbits hop the same distance to the right as to the left". He then typed in the command setup-rabbits 8000 [2 -1] [1 1]. "Now", he said, "the graph won't be symmetrical, it will skew to the right". But to Alan's surprise, when the rabbits hopped, the histogram remained symmetrical, but the purple line representing the average rabbit location (or average population height) moved to the right.

Alan quickly understood the "bug" in his thinking:

"I see. Rabbits hop farther to the right when they hop but they're still just as likely to hop right as left. So, the distribution will remain symmetric, it will just move to the right one unit on average for each step. To make the distribution asymmetric, you need to change the probabilities not the step sizes."
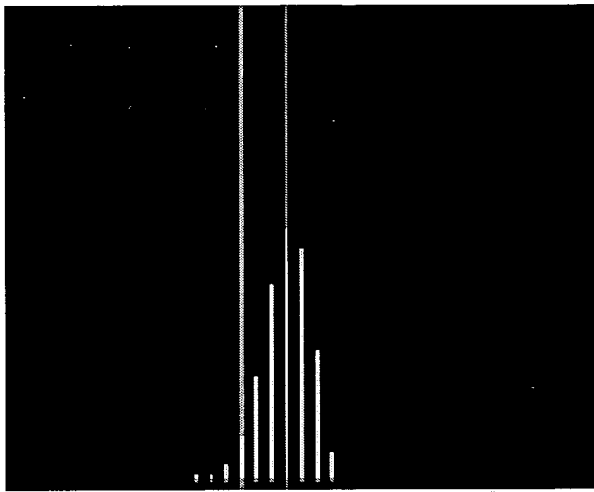
 setup 8000 [2 -2] [3 1] after 10 hops

From the above experiment, Alan concluded:

If children are just as likely to be shorter as they are taller than their parents but when they are taller their difference from their parents is larger, then the population as a whole will slowly get taller. But since we know that height of the population as a whole is symmetrically distributed, children must be equally likely to be taller as they are to be shorter than their parents. I see why normal distributions are so common. Whenever we make a measurement, we're just as likely to make a mistake in one direction as the other, so the resultant distribution of measurements will be normal. Its average will be the true value of the measurement and the spread of the graph will depend on the accuracy of the measurement.

Alan then went on to investigate what would happen to the distributions if he gave the rabbits a much larger "palette" of step sizes. This investigation leads him to more insights about how different micro- level rules produce different probability distributions. When I asked Alan to sum up what he has learned from his modeling experience, he expresses satisfaction that he has gained a deeper understanding of binomial distributions and "what kind of animal a distributions is". But, he also says that perhaps he has not yet "entirely understood normal distributions". He speculates that the normal distribution is "the limit of the binomial distribution as "n" approaches infinity -- when you fill in the gaps between the bars of the binomial". He worries, however, that he doesn"t yet sufficiently understand the connection between the binomial distribution and the formula he remembers for normal distributions: "how does the integral of e -x2/2 come out of this?".

It is important to remember that Alan began the interview with a command of elementary probability and statistics. He was facile with the mathematics of binomial processes and knew the standard parameters and statistics of normal distributions. It was the connection between these two areas of his knowledge that was

missing for Alan. School mathematics, like pre-nineteenth century mathematics keeps these two disciplines separate. Indeed, with limited computational power, it is hard for a school curriculum to make a meaningful connection between the two. Object-based parallel modeling languages bring the two back together. By enabling Alan to give probabilistic rules to the individuals and see the resulting pattern of the ensemble, the StarLogo modeling language enabled him to make these connections. Though Alan has made significant progress in his understanding of distributions, he is not done. His desire to further connect his new understanding to the formula he learned in school exemplifies his appropriation of the Connected Mathematics approach.

7.0 Discussion

It is clear that the source of Lynn and Alan's difficulty with P&S did not lie in a lack of technical proficiency. By most standards, their understanding was advanced. The issue is that their understanding was not well connected. The case studies demonstrate their ability to make these connections and develop rich intuitions in areas, which, in the past, they accommodated by mechanically applying formulae. This supports the Connected Probability project's stance of eschewing the accomodationist view which says "that's the way it is" for P&S, and, instead, viewing P&S as fitting within the larger framework of research in understanding mathematics. Development of intuition and changes in understanding can happen in ways consistent with a broad developmentalist stance. In this discussion, I emphasize the features of the Connected Mathematics therapy that facilitated the development of effective intuitions about normal distributions.

Both Lynn and Alan had strong mathematical backgrounds, had done graduate coursework in mathematics and had excelled. Both of them were very comfortable doing the formal manipulations that constitute the major component of traditional instruction in P&S. Nonetheless, both explicitly expressed considerable discomfort concerning the notion of normal distribution ã the most salient example of the core statistical notion of probability distribution. Regarding the nature of normal distributions, they suffered from epistemological anxiety. The therapy provided in these learning interviews consisted of two main interventions: 1) Validation of the importance of Lynn and Alan's mathematical voice, their personal view of the mathematics and its relation to their own experience. This validation was achieved by explicit avowal at the start of the interview, by open-ended interviews that followed the interviewee's own thinking and by long drawn out interviews that declared of themselves that the interviewee's conceptions were valuable and valued. 2) The use of an object-based parallel modeling language, StarLogo, in which to build computational models of probabilistic phenomena and successively refine them. These supports provided a context in which to work through their epistemological anxiety.

Though, clearly, neither Lynn nor Alan were experts in P&S, their sophisticated mathematics background as well as their formal mastery of P&S techniques would qualify them as experts in the eyes of many educators. These "near-experts" were no less confused about normal distributions than were the novice interviewees. Indeed, instructors of P&S with whom I have shared Lynn's interview have been "appalled at the incoherence of her remarks ", yet, instructors like these, awarded top grades to Lynn in her P&S courses. Interviews such as these cast serious doubt on theories which assert a strong disjunction between novice and expert understanding. Even experts must laboriously map out clarity from a very messy "terrain". By overemphasizing formal operational thought, typical mathematics education allows intellectual accommodation to be an adequate endpoint. In Connected Mathematics, understanding is taken to mean "multiply connected" and not cleanly and uniquely specified. This embraces "messy" conceptions at all levels of expertise in the process of learning. One has never "arrived".

Rather than emphasizing problem solving, Connected Mathematics emphasizes problem posing. In problem solving, the questions are often disembodied and not the learnerês own. Within Connected Mathematics, modeling is both a medium in which learners can formulate their questions in a precise way and a method for building their understanding.

Once Alan posed the problem of the symmetry of the distribution, his reasoning is expressed and refined in interaction with the model he has built. Alan strives to find the connections between his representation of the individual rabbits behaviors and the global pattern that they form. He debugs his model by varying each of the three inputs to the run-rabbits procedure, focusing particularly on the lists of steps and ratios. How will variations in the lengths of the lists and the relative magnitudes of the steps and ratios affect this emergent pattern? He alternates his focus from the micro- level, the behavior of the individual rabbits to the macro- level of the distribution. To better keep track of the change in the global pattern over time, he takes advantage of StarLogoês open-ended programmability and builds representational aids such as the "origin line", the "hopping line" and color-coded classes of rabbits. These representations and tools allow him to debug his original conception that the relative size of the steps is the key to an asymmetric distribution and, instead, to see the relative size of the probabilities as the key factor. By building distributions with probabilistic parts and making them work, Alan makes probability ratios as concrete for himself as step-sizes, and sees how distributions are built out of these concrete building blocks. Extending, enhancing and debugging the computational model itself is essential to the activity of modeling. This distinguishes it from model consumption in which learners are left to tweak the inputs of an expert's model.

In the Connected Probability project, probability distributions are viewed as a kind of emergent phenomena. Rather than manipulate the tokens of formalism, learners are invited to negotiate the interaction of the micro- and macro- levels of probabilistic and statistical phenomena. Through connecting these levels of emergent phenomena, learners understand the "mechanisms" of probability distributions -- how they are assembled from their constituent elements. When these interactions are understood, the global patterns become more than just descriptions to be memorized, they are tools to be used.

Binomial probability distributions are elementary examples of emergent phenomena -- emerging from the behavior of non-interacting individuals. Other probability distributions emerge from more interactive individual behavior. Many natural phenomena can be modeled as systems whose global behavior are not predictable using non-computational mathematics. Such phenomena as the shape of a snowflake, the dynamics of co-extenisve wolf and sheep populations, the economy of a country -- all of these can be modeled as emerging from the interactions of their constituent individuals. Learning to see emergent phenomena -- that is learning to describe the phenomena as arising from the interactions of distributed parts -- is, in itself, a worthy goal for educators. From this perspective, P&S becomes not an end, but an entry point to the world of complex systems.

In the Connected Probability project, the role of technology is as a medium of personal expression and articulation of ideas. Empowering literacy implies being able to say something of your own, rather than just manipulate the texts of others. While models are ubiquitous in the current learning literature, modeling is not. Modeling calls on learners to say something on their own, to be authors of mathematics. This implies the best use of technology is for learners to be model builders not just model consumers. The StarLogo modeling language has specific features which allow for this articulation:

Modeling at the level of objects: StarLogo allowed Alan to create, visualize and modify individual rabbits. Alan was, thus, able to experiment with individual rabbits whose hopping behavior he could track, control, model with his own body. These features of the language enabled Alan to leverage his knowledge of individual behavior to knowledge of the ensemble ã to connect the micro- with the macro-.

Parallelism: The ability to control thousands of rabbits at once gave Alan the experimental apparatus he needed to conduct his investigation in "real time". The ability to experiment with collections of rabbits, to enable sub-populations of rabbits to have different behaviors was crucial to his construction of the distribution.

General purpose programmability: The ability to create his own representations (as opposed to manipulating the parameters of a pre-conceived model) gave Alan the freedom to construct normal distributions in a way that

arose from his own nascent understanding of them. It also enabled him to create computationally active tools, such as the "hopping line". These refinements of his computational model provided greater support for his experimentation and, in turn, led to a refinement of his mental model.

In a Connected Mathematics learning environment, learners are supported in actively connecting areas of their knowledge that have, hitherto, remained separate. Computational platforms, in their ability to bring into being new forms of representation and symbolization, can be powerful tools for making these connections. Object-based parallel modeling provides a new way of coming to understand and articulate understandings of probability and statistics and, in so doing, engage and relieve epistemological anxiety.

Acknowledgments

References

Abelson, H. & diSessa, A. (1980). Turtle Geometry: The Computer as a Medium for Exploring Mathematics. Cambridge, MA: MIT Press.

Abelson, H.. & Goldenberg, P. (1977). Teacherês Guide for Computational Model of Animal Behavior. LOGO Memo NO. 46. Cambridge, MA.

Ball, D. (1990). With an eye on the Mathematical Horizon: Dilemmas of Teaching. Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.

Belenky, M., Clinchy, B., Goldberger, N., & Tarule, J. (1986). *Women's Ways of Knowing.* New York: Basic Books.

Biehler, R. (1994). Cognitive Technologies for Statistics Education. In L. Brunelli & G. Cicchitelli (Eds.), Proceedings of the first scientific meeting of the International Association for Statistical Education.

Borel, E. (1909). *Elments de la Thorie des Probabilitis.* Paris: Hermann et Fils.

Brandes, A. (in press). Elementary School Childrenês Images of Science. in Y. Kafai and M. Resnick (Eds.), *Constructionism in Practice: Rethinking Learning and Its Contexts.*

Brown, S. I. & Walter, M.I. (1990). The art of problem posing (2nd ed.) Hillsdale, NJ.: L. Erlbaum Associates.

Carey, S. (1986). Cognitive Science and Science Education. *American Psychologist,* Vol. 41, No. 10.

Cartwright, N. (1987). In Kruger, L. Daston, L. & Heidelberger, M. (Eds.) The *Probabilistic Revolution.* Vol 2. Cambridge, Ma: MIT Press.

Case, R. (1993). Theories of learning and theories of development, Educational Psychologist, 28 (3).

Chipman, S.F., Krantz, D.H. & Silver, R. (1992). Mathematics anxiety and science careers among able college women. Psychological Sciences, 3, 292-295.

Clement,, J. (1982). Student preconceptions in introductory mechanics, *American Journal of Physics*. 50(66).

Collins, A. & Brown, J.S. (1988). The Computer as a Tool for Learning Through Reflection. In H. Mandl & A. Lesgold (Eds.). *Learning Issues for Intelligent Tutoring Systems* (pp. 1-18). New York: Springer Verlag.

Confrey, J. (1993a). A Constructivist Research Programme Towards the Reform of Mathematics Education. An introduction to a symposium for the Annual Meeting of the *American Educational Research Association*, April 12-17, 1993 in Atlanta, Georgia.

Confrey, J. (1993b). Learning to See Childrenês Mathematics: Crucial Challenges in Constructivist Reform, In K. Tobin (Ed.) *The Practice of Constructivism in Science Education*. Washington, D.C.: American Association for the Advancement of Science. pp. 299-321.

Confrey, J. (in press). Voice and perspective: Hearing epistemological innovation in students' words [Special Issue]. In N. Bednarz, M. Larochelle & J. Desautels (Eds.), *Revue de Sciences de l'education*.

Cosmides, L. & Tooby, J. (1995). Are Humans good Intuitive Statisticians after all? Rethinking some conclusions from the literature on judgment under uncertainty. *Cognition*

Cuoco, A., Goldenberg, P., and Mark, J. (1995). Connecting Geometry with the Rest of Mathematics. In: *Connecting Mathematics across the Curriculum*. The NCTM 1995 Yearbook. House, Peggy A., and Arthur F. Coxford, (Eds.). Reston, VA: National Council of Teachers of Mathematics.

Damarin, S.K. (1995). Gender and Mathematics from a Feminist Standpoint. In W.G. Secada, E. Fennema & L.B. Adajian (Eds.), New directions for Equity in Mathematics Education. New York: Cambridge University Press.

Dantzig, T. (1969). The Bequest of the Greeks. New York: Greenwood Press.

Daston, L. (1987). Rational Individuals versus laws of societr: from Probability to Statistics. In Kruger, L. Daston, L. & Heidelberger, M. (Eds.) The Probabilistic Revolution. Vol 1. Cambridge, Ma: MIT Press.

Dennis, D. & Confrey, J. (1993). The Creation of Continuous Exponents: A Study of the Methods and Epistemology of Alhazen and John Wallis. Paper presented at the annual meeting of the *American Educational Research Association*, April 12-17, 1993 in Atlanta, Georgia.

Diaconis, P. & Bradley, E. (1983).

Dweck, C.S. & Leggett, E.L. (1988). A social-cognitive approach to motivation and personality, *Psychological Review*, 95 (2), 256-273.

Edwards, L. (1995). Microworlds as Representations. in Noss, R., Hoyles, C., diSessa A. and Edwards, L. (Eds.) *Proceedings of the NATO Advanced Technology Workshop on Computer Based Exploratory Learning Environments*. Asilomar, Ca. p. 127-154.

Edwards, L. (1992). A LOGO Microworld for Transformational Geometry. In Hoyles, C. & Noss, R. (Eds.) *Learning Mathematics and Logo*. London: MIT Press. p. 127-155.

Efron, B. (197x).

Feurzeig, W. (1989). A Visual Programming Environment for Mathematics Education. Paper presented at the fourth international conference for Logo and Mathematics Education. Jerusalem, Israel.

Gigerenzer, G. & Hoffrage, U. (1995). How to Improve Bayesian Reasoning without Instruction: Frequency Formats. *Psychological Review*, Vol 102, No. 4.

Gigerenzer, G. (1987). Probabilistic Thinking and the Fight against Subjectivity. In Kruger, L. Daston, L. & Heidelberger, M. (Eds.) *The Probabilistic Revolution*. Vol 2. Cambridge, Ma: MIT Press.

Gilligan, C. (1977). *In a Different Voice: Psychological Theory and Women's Development*. Cambridge, MA: Harvard University Press.

Goldenberg, E. P., Cuoco, A. & Mark, J. (1993). Connected Geometry. *Proceedings of the Tenth International Conference on Technology and Education*. Cambridge, MA, March 21-24, 1993.

Harel, I. (1992). *Children Designers*. Norwood, NJ: Ablex.

Harel, I. & Papert, S. (1990). Software Design as a Learning Environment. *Interactive Learning Environments Journal*. Vol.1 (1). Norwood, NJ: Ablex. p. 41 - 84.

Hoyles, C. (1985). Culture and computers in the mathematics classroom. Inaugural lecture, Bedford Way Paper. London: Institute of Education, University of London.

Hoyles, C. (1991). Computer-based learning environments for Mathematics. In A. Bishop, S. Mellin-Olson, and J. Van Dormolen (Eds.), *Mathematical Knowledge: Its growth Through Teaching*. Dordrecht: Kluwer, 147-172.

Hoyles, C. (1992). A Pedagogy for Mathematical Microworlds. *Educational-Studies-in-Mathematics;* v23 n1 p31-57.

Kahneman, D. (1991). Judgment and Decision Making: A Personal View. *Psychological Science*. vol. 2, no. 3, May 1991. p. 142-145.

Kahneman, D., & Tversky, A. (1982). On the study of Statistical Intuitions. In D. Kahneman, A. Tversky, & D. Slovic (Eds.) *Judgment under Uncertainty: Heuristics and Biases*. Cambridge, England: Cambridge University Press. p. 493-508.

Kahneman, D., & Tversky, A. (1973). On the Psychology of Prediction. *Psychological Review*, 80 (4), pp. 237-251.

Kaput, J. (1989). Notations and Representations. In Von Glaserfeld, E. (Ed.) *Radical Constructivism in Mathematics Education*. Netherlands: Kluwer Academic Publishers. p. 53-74.

Keller, E.F. (1983). *A Feeling for the Organism: The Life and Work of Barbara McClintock*. San Francisco, CA: W.H. Freeman.

Keynes, J.M. (1921). *A Treatise on Probability*. London: MacMillan.

Kline, M. (1972). Mathematical thought from ancient to modern times. New York: Oxford University Press.

Kolmogorov, A.N. (1950). *Foundations of the Theory of Probability*. New York: Chelsea Pub. Co.

Konold, C. (1989). Informal Conceptions of Probability. *Cognition and Instruction,* 6, 59-98.

Konold, C. (1991). Understanding Studentsê beliefs about Probability. In Von Glaserfeld, E. (Ed.) *Radical Constructivism in Mathematics Education*. Netherlands: Kluwer Academic Press. p. 139-156.

Konold, C. (1994). Understanding probability and statistical inference through resampling. In L. Brunelli & G. Cicchitelli (Eds.), Proceedings of the first scientific meeting of the International Association for Statistical Education.

Kruger, L. Daston, L. & Heidelberger, M. (Eds.) (1987). *The Probabilistic Revolution.* Vol 1. Cambridge, Ma: MIT Press.

Lakatos, I. (1976). *Proofs and Refutations.* Cambridge, England: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. In *American Education Research Journal,* spring, vol. 27, no. 1, pp. 29- 63.

Lampert, M. (1991). Managing the Tensions in connecting students inquiry with learning mathematics in school. In *Teaching for Understanding in the age of technology.* D. Perkins, J. Schwartz, & S. Wiske, (Eds.). Cambridge, MA: Harvard University Press.

Larsen, R. & Marx, M. (1986). *An Introduction to Mathematical Statistics and its applications.* Englewood Cliffs, NJ: Prentice Hall.

Leron, U. (1994). Can We Follow What We Preach: Teaching according to constructivist principles. CMESG Procedings, Regina, Sask, Canada.

Leron, U. & Zazkis, R. (1992). Of Geometry, Turtles, and Groups. In Hoyles, C. & Noss, R. (Eds.) *Learning Mathematics and LOGO.* London: MIT Press. p. 319-352.

Leron, U. (1983). Structuring Mathematical Proofs. *American Mathematical Monthly,* Vol. 90, 3, 174-185.

Mandinach, E.B. & Cline, H.F. (1994). *Classroom dynamics. Implementing a technology-based learning environment.* Hillsdale, NJ: Lawrence Erlbaum Associates.

M. McCloskey, Naive theories of motion., In D. Gentner and A. Stevens, (Eds.) *Mental Models,* (Lawrence Erlbaum, Hillsdale, NJ, 1983), pp 299-324.

McLuhan, M. (1964). Understanding Media. New York: McGraw Hill.

Miller, G.A. (1956). The magical number seven plus or minus two: Some limits on our capacity for processing information, *Psychological Review,* 63, 81-97

Minsky, M. (1987a). Form and Content in Computer Science. In ACM Turing Award lectures: the first twenty years, 1966 to 1985. New York: ACM Press ; Reading, Mass.: Addison-Wesley Pub. Co.

Minsky, M. (1987b). *The Society of Mind.* New York: Simon & Schuster Inc.

National Council of Teachers of Mathematics (1991a). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va: NCTM.

National Council of Teachers of Mathematics (1991b). *Professional Standards for Teaching Mathematics.* Reston, Va: NCTM.

Nisbett, R. (1980). *Human Inference: Strategies and Shortcoming of Social Judgment.* Englewood Cliffs, NJ: Prentice-Hall.

Nisbett, R., Krantz, D., Jepson, C., & Kunda, Z. (1983). The Use of Statistical Heuristics in Everyday Inductive

Reasoning. *Psychological Review,* 90 (4), pp. 339-363.

Noss, R. (1988). The computer as a cultural influence in mathematical learning. *Educationasl Studies in Mathematics.* 19, 2.

Noss, R. & Hoyles, C. (1991). Logo and the Learning of Mathematics: Looking Back and Looking Forward. In Hoyles, C. & Noss, R. (Eds.) *Learning Mathematics and Logo.* London: MIT Press. p. 431-468.

Noss, R. (1994). Structure and Ideology in the Mathematics Curriculum. *For the Learning of Mathematics.* 14, 1.

Nozick, R. (1993). *The Nature of Rationality.* Princeton University Press.

OêConnor, C. & Michaels, S. (1993). Aligning Academic Task and Participation Status through Revoicing: Analysis of a Classroom Discourse Strategy. *Anthropology & Education Quarterly.* Vol. 24, No. 4

Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas.* New York: Basic Books.

Papert, S. (1991). Situating Constructionism. In I. Harel & S. Papert (Eds.) *Constructionism.* Norwood, NJ. Ablex Publishing Corp. (Chapter 1). p. 1 - 12.

Papert, S. (1993). *The Childrenês Machine.* New York: Basic Books.

Papert, S. (1972). Teaching Children to be Mathematicians vs. Teaching About Mathematics. *International Journal of Mathematics Education in Science and Technology* Vol. 3, p. 249-262.

Pea, R. (1987). Cognitive Technologies for Mathematics Education. In Schoenfeld, A. (Ed.) *Cognitive Science and Mathematics Education* . Hillsdale, NJ: Erlbaum.

Phillips, J. (1988) *How to Think About Statistics.* New York: W.H. Freeman.

Piaget, J. & Imhelder, B. (1951). *The origin of the idea of chance.* New York: Norton.

Repenning, A. (1993). AgentSheets: A tool for building domain-oriented dynamic, visual environments. Ph.D. dissertation, Dept. of Computer Science, University of Colorado, Boulder.

Resnick, M. (1992). Beyond the Centralized Mindset: Explorations in Massively Parallel Microworlds. Doctoral dissertation, Cambridge, MA: Media Laboratory, MIT.

Resnick, M. (1994). *Turtles, Termites and Traffic Jams.* Explorations in Massively Parallel Microworlds. Cambridge, MA: MIT Press.

Resnick, M. & Wilensky, U. (1995). Beyond the Deterministic, Centralized Mindsets: New Thinking for New Sciences. Paper presented at the annual conference of the American Educational Research Association, Atlanta, GA.

Rotman, B. (1987). *Signifying nothing: the semiotics of zero.* Houndmills,, Basingstoke, Hampshire: Macmillan.

{Rubin, A., Bertram, B., Rosebery, A. & DuMouchel, W. (198x). Getting an Early Start: Using Interactive Graphics to Teach Statistical Concepts in High School. }

Savage, L. (1954). *The Foundations of Statistics.* New York: Wiley.

Scheffler, I. (1991). *In Praise of the Cognitive Emotions.* London: Routledge, Chapman and Hall.

Schoenfeld, A. (1991). On Mathematics as Sense-Making: An Informal Attack on the Unfortunate Divorce of Formal and Informal Mathematics. In Perkins, Segal, & Voss (Eds.) *Informal Reasoning and Education.* Hillsdale, NJ. LEA p. 311-343.

Schwartz, J. & Yerushalmy, M. (1987). The Geometric Supposer: an Intellectual Prosthesis for Making Conjectures. *The College Mathematics Journal,* 18 (1): 58-65.

Sfard. A. & Leron, U. (in preparation). "Just Give Me a Computer and I Will Move the Earth": On the Effect of Programming on Mathematical Problem Solving. Unpublished Manuscript.

Simon, J.L., Atkinson, D.T. & Shevokas, C. (1976). Probability and statistics: Experimental results of a radically different teaching methods, Mathematical Education, November, 733-739.

Simon, J.L. & Bruce, P. (1991). Resampling: A tool for everyday statistical work, Chance: New directions for statistics and computing, 4 (1), 22-32.

Smith, E. & Confrey, J. (in press). Multiplicative Structures and the Development of Logarithms: What was Lost by the Invention of Function? In G. Harel & J. Confrey (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics.* Albany: State University of New York Press. p. 331-360.

Smith, J.P., diSessa, A.A., & Roschelle, J. (1994). Reconceiving Misconceptions: A Constructivist Analysis of Knowledge in Transition. *Journal of the Learning Science,* 3, pp. 115-163.

Steele, C.M. & Aronson, J. (1995). Stereotype Threat and the Intellectual Test Performance of African Americans, Journal of personality and social psychology, 69 (5), 797.

Surrey, J. (1991). Relationship and Empowerment. in Jordan, J., Kaplan, A., Miller, J.B., Stiver, I. & Surrey, J. *Womenês Growth in Connection: Writing from the Stone Center.* New York: The Guilford Press. p. 162-180.

Sheila, T. (1993). Overcoming Math Anxiety. New York: W.W. Norton.

R.K. Thornton and D.R. Sokoloff (1990). Learning motion concepts using real-time, microcomputer-based laboratory tools, *Am.J. Phys.* 58, 858-867

R.K. Thornton and D.R. Sokoloff (in press). Assessing and improving Student learning of Newton's Laws part I: The force and Motion Conceptual Evaluation and active learning laboratory curricula for the First and Second Laws, Am.J. Phys.

Tripp, R.T. (1970). *International Thesaurus of Quotations.* New York: Harper & Row. p. 612.

Turkle S., & Papert, S. (1991). Epistemological Pluralism and Revaluation of the Concrete. In I. Harel & S. Papert (Eds.) *Constructionism.* Norwood NJ. Ablex Publishing Corp. p. 161-192.

Tversky, A. (1973). Availability: A heuristic for judging frequency and probability. Cognitive Psychology, 5, 207-232.

Tversky, A. & Kahneman, D. (1974). Judgment Under Uncertainty: Heuristics and Biases. *Science,* 185, pp. 1124†1131.

Tversky, A. & Kahneman, D. (1980). Causal Schemas in Judgments Under Uncertainty. In M. Fischbein (Ed.), *Progress in Social Psychology.* Hillsdale, NJ: Erlbaum. p. 49-72.

Tversky, A. & Kahneman, D. (1983). Extensional vs. Intuitive Reasoning: The Conjunction Fallacy in Probability Judgment. *Psychological Review*, 90 (4), pp. 293-315.

von Glaserfeld, E. (1987). Preliminaries to any Theory of Representation. In C. Janvier (Eds.) *Problems of Representation in Mathematics Learning and Problem Solving,* Hillsdale, NJ: Erlbaum. p. 215-226.

von Winterfeldt, D. & Edwards, W. (1986). Decision analysis and behavioral research. Cambridge, England, Cambridge University Press.

Walkerdine, V. (1989). Counting girls out. London: Virago.

Wilensky, U. (1991). Abstract Meditations on the Concrete and Concrete Implications for Mathematics Education. In I. Harel & S. Papert (Eds.) *Constructionism.* Norwood NJ.: Ablex Publishing Corp.

Wilensky, U. (1993). Connected Mathematics: Building Concrete Relationships with Mathematical Knowledge. Doctoral dissertation, Cambridge, MA: Media Laboratory, MIT.

Wilensky, U. (1995-a). Learning Probability through Building Computational Models. *Proceedings of the Nineteenth International Conference on the Psychology of Mathematics Education.* Recife, Brazil, July 1995.

Wilensky, (1995-b). Paradox, Programming and Learning Probability: A Case Study in a Connected Mathematics Framework. *Journal of Mathematical Behavior, Volume 14, No 2.*

Wilensky, U. (1995-c). Mathematics as a way of Connecting. Paper presented at the Proceedings of the Boston Forum for Science Education for Future Generations under the auspices of The American Academy of Arts and Sciences.

Wilensky, U. (in press). Making Sense of Probability through Paradox and Programming. in Y. Kafai and M. Resnick (Eds.), Constructionism in Practice: Rethinking Learning and Its Contexts.

Wilensky, U. (forthcoming). GPCEEãan Extensible Microworld for Exploring Micro- and Macro- Views of Gases. Manuscript submitted for publication.